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On the noncommutative geometry of the endomorphism algebra of a vector bundle

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Abstract

In this paper we investigate some aspects of the noncommutative differential geometry based on derivations of the algebra of endomorphisms of an oriented complex hermitian vector bundle. We relate it, in a natural way, to the geometry of the underlying principal bundle, we introduce on it a notion of metric and we study the cohomology of its complex of noncommutative differential forms. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [5], it was shown that the noncommutative geometry of the algebra of endomorphisms of an oriented complex hermitian vector bundle is very much like the ordinary geometry of the associated $SU(n)$-principal bundle. In particular, from the point of view of connections, this noncommutative algebra gives us interesting relations with the canonical Atiyah Lie algebroid associated with this oriented vector bundle.

In this paper, we would like to proceed in the study of the noncommutative differential calculus of this endomorphism algebra. In particular, we would like to make closer relations with the ordinary geometry of the principal bundle. Using ordinary technics in algebraic geometry, we will study the cohomology of this noncommutative differential calculus.

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The notations we will use in this paper are those of Dubois-Violette and Masson [5], which we recall in the next section, with the main results of Dubois-Violette and Masson [5].

2. Notations and useful previous results

The noncommutative differential calculus based on derivations has been introduced and studied in [2,5,6]. For the matrix algebra $M_n(\mathbb{C})$, its complex of differential forms is $\Omega_{\mathrm{Der}}(M_n(\mathbb{C})) = M_n(\mathbb{C}) \otimes \bigwedge sl(n,\mathbb{C})^*$. The differential d' is given by the Lie algebra structure of $sl(n,\mathbb{C})$, if we consider this complex as a complex for Lie cohomology with values in $M_n(\mathbb{C})$ for the adjoint representation.

For a matrix-valued functions algebra $\mathfrak{B} = C^\infty(P) \otimes M_n(\mathbb{C})$, where P is a manifold, there is a canonical decomposition $\Omega_{\mathrm{Der}}(\mathfrak{B}) = \Omega(P) \otimes \Omega_{\mathrm{Der}}(M_n(\mathbb{C}))$ where $\Omega(P)$ is the ordinary de Rham complex of differential forms on P. The differential is $d + d'$ where d is the ordinary de Rham differential on $\Omega(P)$ [4].

Now, let M be a regular finite-dimensional smooth manifold. We denote by E an oriented complex hermitian vector bundle of rank n over M, by P its $SU(n)$-principal frame bundle, and we introduce, as in [5], the algebra $\mathfrak{A}$ of endomorphisms of E, which is the set of sections of $\mathrm{End}(E)$.

We denote by $(\Omega_{\mathrm{Der}}(\mathfrak{A}), \hat{d})$ the noncommutative differential calculus based on derivations associated to $\mathfrak{A}$. In [5], we proved that we can associate (canonically and in an injective way) to any $SU(n)$-connection on E a noncommutative 1-form $\alpha \in \Omega^1_{\mathrm{Der}}(\mathfrak{A})$. Locally, on an open subset U of M over which E is trivial, any derivation $\mathcal{X}$ of $\mathfrak{A}$ can be decomposed as $\mathcal{X}_{\mathrm{loc}} = X + ad_\gamma$, with X an ordinary vector field on U and γ a traceless matrix-valued function on U. In this trivialization, α can be written as $\alpha_{\mathrm{loc}}(\mathcal{X}_{\mathrm{loc}}) = A(X) - \gamma$ where A is the local connection 1-form of the connection on E. In a second trivialization over a second open subset U' of M, one has $\mathcal{X}_{\mathrm{loc}} = X + ad_{\gamma'}$, and on $U \cap U'$, one has $\gamma' = g^{-1}\gamma g + g^{-1}(Xg)$ where $g : U \cap U' \to SU(n)$ is the corresponding transition function of E [5].

3. Some relations between $\Omega_{\mathrm{Der}}(\mathfrak{A})$ and $\Omega(P)$

In this section we would like to give some structure properties of $\Omega_{\mathrm{Der}}(\mathfrak{A})$ which relates it to the ordinary differential calculus $\Omega(P)$ of P.

Let us denote by $\mathcal{F}(P)$ the (commutative) algebra of smooth functions on P and by $A \mapsto A^v$ the map which sends any $A \in \mathfrak{su}(n)$ into the associated vertical vector field on P.

Let us introduce the algebra $\mathfrak{B} = \mathcal{F}(P) \otimes M_n(\mathbb{C})$ of matrix-valued functions on P. Denote by $(\Omega_{\mathrm{Der}}(\mathfrak{B}), \hat{d}) = (\Omega(P) \otimes \Omega_{\mathrm{Der}}(M_n(\mathbb{C})), d + d')$ its differential calculus based on derivations. It is easy to see that $\mathcal{G} = \{A^v + ad_A \ / \ A \in \mathfrak{su}(n)\}$ is a Lie subalgebra of $\mathrm{Der}(\mathfrak{B})$ isomorphic to $\mathfrak{su}(n)$. This Lie subalgebra defines a Cartan operation of $\mathfrak{su}(n)$ on $\Omega_{\mathrm{Der}}(\mathfrak{B})$, whose basic subalgebra we denote by $\Omega_{\mathrm{Der,Bas}}(\mathfrak{B})$.

Proposition 1. $\Omega_{\mathrm{Der}}(\mathfrak{A}) = \Omega_{\mathrm{Der,Bas}}(\mathfrak{B})$.

Proof. The proof is based on results on noncommutative quotient manifolds we introduced and studied in [7]. First, notice that $\mathfrak{A}$, as a set of section of an associated bundle of P, can be considered as the algebra $\{a \in \mathfrak{B}/A^v a + [A, a] = 0\ \forall A \in \mathfrak{su}(n)\}$. Now, define as in [7]

$$\mathcal{Z}_{\mathrm{Der}}(\mathfrak{A}) = \{\mathfrak{X} \in \mathrm{Der}(\mathfrak{B})/\mathfrak{X}\mathfrak{A} = 0\},$$
$$\mathcal{N}_{\mathrm{Der}}(\mathfrak{A}) = \{\mathfrak{X} \in \mathrm{Der}(\mathfrak{B})/\mathfrak{X}\mathfrak{A} \subset \mathfrak{A}\}.$$

Then looking locally (in a trivialization of P) at the derivations of $\mathfrak{B}$, one sees that

$$\mathcal{Z}(\mathfrak{A}) = \mathfrak{A} \cap \mathcal{Z}(\mathfrak{B}),$$
$$\mathrm{Der}(\mathfrak{A}) = \mathcal{N}_{\mathrm{Der}}(\mathfrak{A})/\mathcal{Z}_{\mathrm{Der}}(\mathfrak{A}),$$
$$\mathfrak{A} = \{b \in \mathfrak{B}/\mathfrak{X}b = 0\ \forall\ \mathfrak{X} \in \mathcal{Z}_{\mathrm{Der}}(\mathfrak{A})\},$$

where $\mathcal{Z}(\mathfrak{A})$ and $\mathcal{Z}(\mathfrak{B})$ are the center of the algebras $\mathfrak{A}$ and $\mathfrak{B}$, respectively. This makes $\mathfrak{A}$ into a noncommutative quotient manifold algebra of $\mathfrak{B}$ in the sense of Masson [7]. In order to prove the proposition, using Proposition V.1 in [7] and the fact that $\underline{\Omega}_{\mathrm{Der}}$ and Ω_{Der} coincide in this context [5], it remains to show that the $\mathcal{Z}(\mathfrak{B})$-module induced by $\mathcal{N}_{\mathrm{Der}}(\mathfrak{A})$ in $\mathrm{Der}(\mathfrak{B})$ is $\mathrm{Der}(\mathfrak{B})$ itself. As before, using local expressions of derivations, this can be checked easily. □

Let us give an interesting example. Consider an $SU(n)$-connection on P. Denote by ω its 1-form on P. It was shown in [5] that there exists a corresponding noncommutative 1-form $\alpha \in \Omega_{\mathrm{Der}}(\mathfrak{A})$. From Proposition 1, this form comes from a basic 1-form in $\Omega_{\mathrm{Der,Bas}}(\mathfrak{B})$, which is nothing but $\omega - \mathrm{i}\theta$, where $\theta \in \Omega^1_{\mathrm{Der}}(M_n(\mathbb{C}))$ is the canonical 1-form defined in [3] by $\mathrm{i}\theta(ad_\gamma) = \gamma - 1/n\,\mathrm{Tr}(\gamma)\mathbb{1}$ for any $\gamma \in M_n(\mathbb{C})$. The basicity of this 1-form is a consequence of properties of ω and $\mathrm{i}\theta$, in particular the equivariance of ω.

Now, notice that the commutative algebra $\mathcal{F}(M)$ of smooth functions on M and its de Rham complex $\Omega(M)$ are the basic subalgebra of $\mathcal{F}(P)$ and the basic subcomplex of $\Omega(P)$ for the operation of $\mathfrak{su}(n)$ induced by $A \mapsto A^v$. This operation is itself the restriction of the operation of $\mathfrak{su}(n)$ considered previously. Then, from this point of view, $\Omega_{\mathrm{Der}}(\mathfrak{A})$ is a natural generalization of $\Omega(M)$ containing informations on P.

Moreover, this construction fits perfectly with the notion of noncommutative integration. It was shown in [3] that such a notion exists on the noncommutative differential calculus $\Omega_{\mathrm{Der}}(M_n(\mathbb{C}))$. We denote by $\omega \in \Omega^{n^2-1}_{\mathrm{Der}}(M_n(\mathbb{C})) \mapsto \int_{\mathrm{nc}} \omega \in \mathbb{C}$ this noncommutative integral. Recall that it is defined by the following procedure. Let $\{\partial_k = ad_{iE_k}\}_{k=1,\dots,n^2-1}$ denotes a basis of $\mathrm{Der}(M_n(\mathbb{C})) \simeq sl(n, \mathbb{C})$, where $\{E_k\}_{k=1,\dots,n^2-1}$ is a basis of hermitean (traceless) matrices of $sl(n, \mathbb{C})$. Denote by $\{\theta^l\}$ the dual basis of $\{\partial_k\}$ in $sl(n, \mathbb{C})^*$, and $g_{kl} = 1/n\,\mathrm{Tr}(E_k E_l)$ the natural metric on $\mathrm{Der}(M_n(\mathbb{C}))$. Then the noncommutative $(n^2 - 1)$-form $\sqrt{g}\theta^1 \cdots \theta^{n^2-1}$ is a volume form, where $\sqrt{g}$ is the square root of the determinant of the metric. Any noncommutative $(n^2 - 1)$-form $\omega \in \Omega^{n^2-1}_{\mathrm{Der}}(M_n(\mathbb{C}))$ can be uniquely written $\omega = M\sqrt{g}\theta^1 \cdots \theta^{n^2-1}$ and we define $\int_{\mathrm{nc}} \omega = 1/n\,\mathrm{Tr}(M)$, which can be proven to be independant of the choice of the basis $\{E_k\}$.

Integrating only the noncommutative part of any noncommutative form on $\mathfrak{B}$, we get a map

$$\begin{aligned}\Omega_{\mathrm{Der}}^{p,n^2-1}(\mathfrak{B}) &\to \Omega^p(P)\\ \omega &\mapsto \int_{\mathrm{nc}} \omega,\end{aligned}$$

which satisfies:

Proposition 2.

(1) *If $\omega \in \Omega_{\mathrm{Der}}^{p,n^2-1}(\mathfrak{B})$ is horizontal (resp. invariant), then $\int_{\mathrm{nc}} \omega \in \Omega^p(P)$ is horizontal (resp. invariant) for the two operations defined above.*

(2) *Considering basic elements, this map defines a canonical noncommutative integration "along the (noncommutative) fiber" $\Omega_{\mathrm{Der}}(\mathfrak{A}) \to \Omega(M)$.*

(3) *This noncommutative integral is compatible with the differentials:*

$$\int_{\mathrm{nc}} \hat{d}\omega = \mathrm{d} \int_{\mathrm{nc}} \omega.$$

(4) *This induces maps in cohomologies:*

$$\int_{\mathrm{nc}\,*} : H^r(\Omega_{\mathrm{Der}}(\mathfrak{B}), \hat{d}) \to H^{r-(n^2-1)}(P),$$

$$\int_{\mathrm{nc}\,*} : H^r(\Omega_{\mathrm{Der}}(\mathfrak{A}), \hat{d}) \to H^{r-(n^2-1)}(M).$$

Proof. (1) and (3) are straightforward computations using various properties of the trace and the volume form in the definition of the noncommutative integration over $\Omega_{\mathrm{Der}}(M_n(\mathbb{C}))$, and properties of the differentials and the two operations.

(2) and (4) are immediate consequences of (1) and (3). □

The cohomology groups involved in (4) will be described in the Section 6 where some results on the cohomology of $\Omega_{\mathrm{Der}}(\mathfrak{A})$ are given.

The different relations between the various differential calculi can be summarized in the following diagram:

$$\begin{array}{ccc}\Omega_{\mathrm{Der}}(\mathfrak{A}) & \xrightarrow[\text{elements}]{\text{basic}} & \Omega_{\mathrm{Der}}(\mathfrak{B}) \\ \Big\downarrow {\scriptstyle \int_{\mathrm{nc}}} & & \Big\downarrow {\scriptstyle \int_{\mathrm{nc}}} \\ \Omega(M) & \xrightarrow[\text{elements}]{\text{basic}} & \Omega(P)\end{array}$$

4. The local point of view

It is possible to look at this integration from a local point of view. Let U be an open subset of a chart of M over which E is trivialized. Denote by $(x^1, \dots, x^m)$ some coordinates on U. Then the algebra $\mathfrak{A}$ looks locally (restricting elements of $\mathfrak{A}$ to U) as the algebra $\mathcal{F}(U) \otimes M_n(\mathbb{C})$. We will use for the matrix part the basis $\{E_k\}$, $\{\partial_k\}$ and $\{\theta^k\}$ introduced before. Any noncommutative form ω can be locally decomposed as

$$\omega_{\mathrm{loc}} = \sum_{p,q} \widetilde{\omega}_{\mu_1 \cdots \mu_p, k_1 \cdots k_q} \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \theta^{k_1} \cdots \theta^{k_q},$$

with $\widetilde{\omega}_{\mu_1 \cdots \mu_p, k_1 \cdots k_q} \in \mathcal{F}(U) \otimes M_n(\mathbb{C})$ and where all the 1-form $\mathrm{d}x^\mu$ and θ^k anticommute between themselves. The basis $\{\theta^k\}$ chosen for the purely noncommutative part is not well adapted to study transition relations between two open subsets U and U' of M. In particular, in this basis, it is impossible to get some tractable transition relations between the local matrix-valued functions $\widetilde{\omega}_{\mu_1 \cdots \mu_p, k_1 \cdots k_q}$ and $\widetilde{\omega}'_{\mu_1 \cdots \mu_{p'}, k_1 \cdots k_{q'}}$ defined over U and U', respectively.

There is a better basis defined in the following way. Introduce an $SU(n)$-connection on E whose local connection 1-forms are A on U and A' on U'. Define $n^2 - 1$ noncommutative 1-forms on U, with values in the center of the algebra, by

$$\alpha^r = A^r - \mathrm{i}\theta^r,$$

where $A = A^r \otimes E_r$ on U. The notation α^r comes from the fact that those 1-forms are the local components of the noncommutative 1-form α associated to the connection on E: over U, one has $\alpha_{\mathrm{loc}} = \alpha^r E_r$. Because α is a well-defined global form, the local 1-forms α^r have good transition relations. Denote by $g : U \cap U' \to SU(n)$ the transition function of E associated to U and U', and introduce the $n^2 - 1$ by $n^2 - 1$ matrix-valued function G by $g^{-1} E_k g = G^l_k E_l$. Denote by α'^s the $n^2 - 1$ noncommutative 1-forms on U' defined by $\alpha'^s = A'^s - \mathrm{i}\theta^s$. Then it is easy to show that we have the homogeneous transition relations

$$\alpha'^s = G^s_r \alpha^r$$

over $U \cap U'$. Now, any noncommutative form ω can be locally decomposed over U as

$$\omega_{\mathrm{loc}} = \sum_{p,q} \omega_{\mu_1 \cdots \mu_p, r_1 \cdots r_q} \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \alpha^{r_1} \cdots \alpha^{r_q},$$

where as before, $\omega_{\mu_1 \cdots \mu_p, r_1 \cdots r_q} \in \mathcal{F}(M) \otimes M_n(\mathbb{C})$. It is now possible to compute the transition relations between the $\omega_{\mu_1 \cdots \mu_p, r_1 \cdots r_q}$ and $\omega'_{\mu_1 \cdots \mu_{p'}, r_1 \cdots r_{q'}}$. One gets the simple relation

$$\omega'_{\mu_1 \cdots \mu_p, r_1 \cdots r_q} G^{r_1}_{s_1} \cdots G^{r_q}_{s_q} = g^{-1} \omega_{\mu_1 \cdots \mu_p, s_1 \cdots s_q} g.$$

In order to simplify the analysis, we have assumed that we do not need to change the local coordinates $(x^1, \dots, x^m)$. Indeed, the relations one would obtain when a change of coordinates is performed, are the usual ones.

Consider now the particular local $(n^2 - 1)$-forms

$$\omega^{\alpha}_{\mathrm{loc}} = \alpha^1 \cdots \alpha^{n^2-1}.$$

Using the fact that $\det(G) = 1$, it is easy to see that those local forms can be glued together to define a global form $\omega^{\alpha} \in \Omega^{n^2-1}_{\mathrm{Der}}(\mathfrak{A})$. Notice that the highest degree in the noncommutative part does not depend on the connection. This global form plays the role of a noncommutative volume form in the sense that any form $\omega \in \Omega^{r}_{\mathrm{Der}}(\mathfrak{A})$ can be written

$$\omega = \left(\int_{\mathrm{nc}} \omega \right) \omega^{\alpha} + \eta.$$

In this expression, η is locally the sum of terms which are not of bidegree $(r - n^2 + 1, n^2 - 1)$ or such that their trace are zero. We have trivially that

$$\int_{\mathrm{nc}} \omega^{\alpha} = \mathbb{1} \in \mathcal{F}(M).$$

To compute the noncommutative integral of ω, one has just to decompose it locally in the form $\omega_{\mathrm{loc}} = \rho_{\mathrm{loc}} \omega^{\alpha}_{\mathrm{loc}} + \eta_{\mathrm{loc}}$, with η_{loc} as before, and take the trace of ρ_{loc}. This trace defines a global ordinary form on M because ρ_{loc} defined on U and ρ'_{loc} defined on U' are related by the adjoint representation. The result does not depend on the choice of the $SU(n)$-connection because only the term of bidegree $(0, n^2 - 1)$ in $\omega^{\alpha}_{\mathrm{loc}}$ is important.

5. Riemannian structure and Hodge operator

The notion of riemannian structure we would like to introduce on this noncommutative geometry looks very similar to the one that has been defined in [6]. Here, a riemannian structure is a symmetric $\mathcal{Z}(\mathfrak{A})$-bilinear mapping

$$h : \mathrm{Der}(\mathfrak{A}) \otimes_{\mathcal{Z}(\mathfrak{A})} \mathrm{Der}(\mathfrak{A}) \to \mathcal{Z}(\mathfrak{A}),$$

which is nondegenerate in the sense that the map $\mathrm{Der}(\mathfrak{A}) \to \Omega^1_{\mathrm{Der}}(\mathfrak{A})$ defined by $\mathcal{X} \mapsto [\mathcal{Y} \mapsto h(\mathcal{X}, \mathcal{Y})]$ is injective. We will call h a metric on $\mathrm{Der}(\mathfrak{A})$.

Let us look at h in a local trivialization of E over an open subet U of M. With $\mathcal{X}_{\mathrm{loc}} = X^{\mu}\partial_{\mu} + \gamma^k ad_{E_k}$ and $\mathcal{Y}_{\mathrm{loc}} = Y^{\nu}\partial_{\nu} + \eta^l ad_{E_l}$, one has

$$h_{\mathrm{loc}}(\mathcal{X}_{\mathrm{loc}}, \mathcal{Y}_{\mathrm{loc}}) = h_{\mu\nu} X^{\mu} Y^{\nu} + h_{\mu l} X^{\mu} \eta^l + h_{k\nu} \gamma^k Y^{\nu} + h_{kl} \gamma^k \eta^l,$$

where the $h_{\mu\nu}, h_{\mu l}, h_{k\nu}, h_{kl}$ are local functions on U. The transition relations between those functions are obtained using the transition relations between the local expressions of the derivations. Denote by (h^{rs}) the matrix inverse of (h_{kl}). Then the quantities

$$A^r_{\mu} = -h^{rs} h_{r\mu}$$

have the transitions' relations of the components of a local $SU(n)$-connection 1-form on E, and the quantities

$$h^M_{\mu\nu} = h_{\mu\nu} - h_{kl} A^k_\mu A^l_\nu$$

define a riemannian metric h^M on M. A natural candidate for the h_{kl} are

$$h_{kl} = \frac{1}{n} \operatorname{Tr}(E_k E_l),$$

which is the natural metric on the noncommutative part. We can associate to the connection defined by the A^k_μ the noncommutative 1-form α, and we can then write the metric h as

$$h(\mathcal{X}, \mathcal{Y}) = h^M(X, Y) + \frac{1}{n} \operatorname{Tr}(\alpha(\mathcal{X})\alpha(\mathcal{Y})).$$

This situation is very similar to the one encountered in Kaluza–Klein theories.

We can now look at this metric from a dual point of view. In this case, this is a bimodule map $\Omega^1_{\mathrm{Der}}(\mathfrak{A}) \otimes_{\mathfrak{A}} \Omega^1_{\mathrm{Der}}(\mathfrak{A}) \to \mathfrak{A}$, defined locally by

$$h_{\mathrm{loc}}(\rho_{\mathrm{loc}}, \eta_{\mathrm{loc}}) = h^{M\mu\nu}(\rho_\mu \eta_\nu + \rho_k A^k_\mu \eta_\nu + \rho_\mu A^l_\nu \eta_l) + \rho_k (h^{kl} + h^{M\mu\nu} A^k_\mu A^l_\nu) \eta_l,$$

where $\rho, \eta \in \Omega^1_{\mathrm{Der}}(\mathfrak{A})$ are written locally as $\rho_{\mathrm{loc}} = \rho_\mu \, \mathrm{d}x^\mu + \rho_k \mathrm{i}\theta^k$ and $\eta_{\mathrm{loc}} = \eta_\nu \, \mathrm{d}x^\nu + \eta_l \mathrm{i}\theta^l$.

Suppose now that the base $\{E_k\}$ of hermitean traceless matrices is orthonormal for the natural metric: $h_{kl} = \delta_{kl}$. Then an easy computation show that the $n^2 - 1$ local noncommutative 1-form α^r introduced in Section 4 satisfy

$$h(\alpha^r, \alpha^s) = \delta^{rs}.$$

With these forms, it is possible to define a Hodge star operator

$$* : \Omega^k_{\mathrm{Der}}(\mathfrak{A}) \to \Omega^{m+n^2-1-k}_{\mathrm{Der}}(\mathfrak{A}).$$

Locally, for $\omega_{\mathrm{loc}} = \omega_{\mu_1 \cdots \mu_p, r_1 \cdots r_q} \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \alpha^{r_1} \cdots \alpha^{r_q}$ with $p + q = k$, define

$$\begin{aligned}(*\omega)_{\mathrm{loc}} = {} & (-1)^{q(m-p)} \frac{1}{p!q!} \sqrt{\det(h^M)} \, \omega_{\mu_1 \cdots \mu_p, r_1 \cdots r_q} \varepsilon_{\nu_1 \cdots \nu_m} \varepsilon_{s_1 \cdots s_{n^2-1}} \\ & \times h^{M\mu_1\nu_1} \cdots h^{M\mu_p\nu_p} \delta^{r_1 s_1} \cdots \delta^{r_q s_q} \, \mathrm{d}x^{\nu_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{\nu_m} \alpha^{s_{q+1}} \cdots \alpha^{s_{n^2-1}}.\end{aligned}$$

A straightforward computation shows that this formula define a well-global $(m+n^2-1-k)$-form over $\mathfrak{A}$, and that the star operator satisfies

$$**\,\omega = (-1)^{(m+n^2-1-(p+q))(p+q)} \omega.$$

This star operator gives rise to a scalar product on noncommutative forms by

$$\langle \omega, \omega' \rangle = \int_M \int_{\mathrm{nc}} \omega * \omega'.$$

One can then define an adjoint operator $\hat{\delta}$ to $\hat{d}$ as usual. A straightforward computation shows that $\hat{\delta}\alpha = 0$ where α is associated to the A^k_μ.

6. Some results on the cohomology of $\Omega_{\mathrm{Der}}(\mathfrak{A})$

In the case when $\mathfrak{A}$ is a tensor product $\mathfrak{A} = \mathcal{F}(M) \otimes M_n(\mathbb{C})$, the cohomology of $\Omega_{\mathrm{Der}}(\mathfrak{A}) = \Omega(M) \otimes \Omega_{\mathrm{Der}}(M_n(\mathbb{C}))$ is known [4]. It is just the tensor product of the cohomology of $\Omega(M)$ (the de Rham cohomology) and the cohomology of $\Omega_{\mathrm{Der}}(M_n(\mathbb{C}))$ (we will detail this last cohomology in the following).

In the general case, we will study the cohomology of $\Omega_{\mathrm{Der}}(\mathfrak{A})$ using a slight modification of standard constructions in algebraic topology [1]. We will only emphasize the points which are different from the standard construction in [1].

Let U be an open subset of M such that the restriction of $\mathrm{End}(E)$ to U is trivial. We make a choice of trivializations for any such open subset and we denote by $\mathfrak{A}(U)$ the trivialization of the restriction to U of the algebra $\mathfrak{A}$. Then one has $\mathfrak{A}(U) \simeq \mathcal{F}(U) \otimes M_n(\mathbb{C})$. Denote by $g_{UV} : U \cap V \to SU(n)$ the transition functions.

Consider now the presheaf $\mathcal{F}$ defined by $U \mapsto \Omega_{\mathrm{Der}}(\mathfrak{A}(U))$ where U is any open subset of M which trivializes $\mathrm{End}(E)$. For $V \subset U$, the inclusion map is defined to be

$$\begin{aligned} i_U^V : \Omega_{\mathrm{Der}}(\mathfrak{A}(U)) &\to \Omega_{\mathrm{Der}}(\mathfrak{A}(V)) \\ \omega &\mapsto (\omega_{\restriction V})^{g_{UV}}, \end{aligned}$$

where $(\omega_{\restriction V})^{g_{UV}}$ is the action of g_{UV} (change of trivialization) on the restriction of ω to V. If $\omega = a_0 \hat{d}a_1 \cdots \hat{d}a_p$, one has

$$\omega^{g_{UV}} = (g_{UV}^{-1} a_0 g_{UV}) \hat{d}(g_{UV}^{-1} a_1 g_{UV}) \cdots \hat{d}(g_{UV}^{-1} a_p g_{UV})$$

This action commutes with d.

Now, let us take a good cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ of M indexed by an ordered set I and such that over any U_α, $\mathrm{End}(E)$ is trivialized. For convenience, on any $U_{\alpha_1 \cdots \alpha_q} = U_{\alpha_1} \cap \cdots \cap U_{\alpha_q}$ the trivialization is chosen to be the restriction of the trivialization of U_{α_q}.

We now define a noncommutative version of the double Čech–de Rham complex associated with this presheaf. For $p \geq 0$ and $q \geq 0$, consider the vector spaces

$$C^{p,q}(\mathfrak{U}, \mathcal{F}) = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q_{\mathrm{Der}}(\mathfrak{A}(U_{\alpha_0 \cdots \alpha_p})).$$

An element $\omega \in C^{p,q}(\mathfrak{U}, \mathcal{F})$ is a collection of noncommutative q-form in $\omega_{\alpha_0 \cdots \alpha_p} \in \Omega^q_{\mathrm{Der}}(\mathfrak{A}(U_{\alpha_0 \cdots \alpha_p}))$.

Define the differential $\delta : C^{p,q}(\mathfrak{U}, \mathcal{F}) \to C^{p+1,q}(\mathfrak{U}, \mathcal{F})$ by

$$(\delta\omega)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{i=0}^{p} (-1)^i \omega_{\alpha_0 \cdots \alpha_{i-1} \alpha_{i+1} \cdots \alpha_{p+1}} + (-1)^{p+1} \omega_{\alpha_0 \cdots \alpha_p}^{g_{\alpha_p \alpha_{p+1}}}.$$

Using the properties of the transition functions it is easy to verify that $\delta^2 = 0$. The noncommutative differential $\hat{d}$ is of degree $(0, 1)$ on this double complex and satisfies $\hat{d}\delta = \delta\hat{d}$. On the total complex of this double complex, we introduce the differential $D = \delta + (-1)^p \hat{d}$, with $D^2 = 0$.

For $p = -1$, define $C^{-1,q}(\mathfrak{U}, \mathcal{F})$ to be $\Omega^q_{\mathrm{Der}}(\mathfrak{A})$, and $\delta : C^{-1,q}(\mathfrak{U}, \mathcal{F}) \to C^{0,q}(\mathfrak{U}, \mathcal{F})$ to be the restriction map. Then $\delta^2 = 0$ also holds on $C^{-1,q}(\mathfrak{U}, \mathcal{F})$.

Lemma 3. *The following sequence is exact:*

$$0 \longrightarrow C^{-1,*}(\mathfrak{U}, \mathcal{F}) \xrightarrow{\delta} C^{0,*}(\mathfrak{U}, \mathcal{F}) \xrightarrow{\delta} C^{1,*}(\mathfrak{U}, \mathcal{F}) \xrightarrow{\delta} \cdots$$

Proof. The exactitude at $C^{-1,*}(\mathfrak{U}, \mathcal{F})$ is trivial. Because $\delta^2 = 0$, one has only to show that if $\omega \in C^{p,*}(\mathfrak{U}, \mathcal{F})$, with $\delta\omega = 0$, then there exists $\eta \in C^{p-1,*}(\mathfrak{U}, \mathcal{F})$ such that $\delta\eta = \omega$. Introduce $\{\rho_\alpha\}$ a partition of unity subordinate to the good cover $\mathfrak{U}$. For $\alpha_0 < \cdots < \alpha_{p-1}$, define

$$\eta_{\alpha_0\cdots\alpha_{p-1}} = \sum_{\substack{\alpha<\alpha_{p-1} \\ \alpha\neq\alpha_0,\dots,\alpha_{p-2}}} \rho_\alpha \omega_{\alpha\alpha_0\cdots\alpha_{p-1}} + \sum_{\alpha>\alpha_{p-1}} \rho_\alpha \omega_{\alpha\alpha_0\cdots\alpha_{p-1}}^{g_{\alpha\alpha_{p-1}}},$$

where to simplify this expression, we make use of the notation $\omega_{\dots\alpha_i\cdots\alpha_j\cdots} = -\omega_{\dots\alpha_j\dots\alpha_i\dots}$. Note that $\rho_\alpha\omega_{\alpha\alpha_0\cdots\alpha_{p-1}} \in \Omega_{\mathrm{Der}}(\mathfrak{A}(U_{\alpha\alpha_0\cdots\alpha_{p-1}}))$. A straightforward computation shows then that with $\delta\omega = 0$, one has $\delta\eta = \omega$. □

Using general arguments on double complexes, this lemma gives us a noncommutative version of the generalized Mayer–Vietoris principle:

Corollary 4. *The cohomology of $(\Omega_{\mathrm{Der}}(\mathfrak{A}), \hat{d})$ is the cohomology of the total complex $(C(\mathfrak{U}, \mathcal{F}), D)$.*

Consider now the spectral sequence $\{E_r, d_r\}$ associated with the double complex $C(\mathfrak{U}, \mathcal{F})$, induced by the filtration

$$F^p C(\mathfrak{U}, \mathcal{F}) = \oplus_{s\geq p} \oplus_{q\geq 0} C^{s,q}(\mathfrak{U}, \mathcal{F}).$$

Then by standard argument, one knows that the first term of this spectral sequence is

$$E_1^{p,q} = H_{\hat{d}}^{p,q} = C^p(\mathfrak{U}, \mathcal{H}^q),$$

where $\mathcal{H}^q$ is the presheaf $\mathcal{H}^q(U) = H^q(\Omega_{\mathrm{Der}}(\mathfrak{A}(U)), \hat{d})$ and that the second term is

$$E_2^{p,q} = H_\delta^p(\mathfrak{U}, \mathcal{H}^q).$$

This spectral sequence converges to $H(C(\mathfrak{U}, \mathcal{F}), D) = H(\Omega_{\mathrm{Der}}(\mathfrak{A}), \hat{d})$.

The cohomology groups $H^*(\Omega_{\mathrm{Der}}(\mathfrak{A}(U)), \hat{d})$ have been computed [2–4]. When U is diffeomorphic to $\mathbb{R}^m$, one has

$$H^*(\Omega_{\mathrm{Der}}(\mathfrak{A}(U)), \hat{d}) = H^*(\Omega_{\mathrm{Der}}(M_n(\mathbb{C})), d')$$

and $H^*(\Omega_{\mathrm{Der}}(M_n(\mathbb{C})), d')$ is isomorphic to the cohomology of the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. Then $H^*(\Omega_{\mathrm{Der}}(M_n(\mathbb{C})), d') = \left(\bigwedge \mathfrak{sl}(n, \mathbb{C})^*\right)_{\mathrm{Inv}}$, where Inv denotes the invariant elements for the canonical Lie derivation. This algebra can also be considered as the graded commutative algebra freely generated by c^n_{2r-1} in degree $2r-1$ with $r \in \{2, 3, \dots, n\}$ and where the c^n_{2r-1} are the primitive elements of $\left(\bigwedge \mathfrak{sl}(n, \mathbb{C})^*\right)_{\mathrm{Inv}}$.

Any element in $\left(\bigwedge \mathfrak{sl}(n, \mathbb{C})^*\right)_{\mathrm{Inv}}$ is then invariant by the action of $SU(n)$. So, the inclusion map is the identity and then the presheaf $\mathcal{H}^q$ is a constant presheaf. The cohomology of E_1 reduces to the Čech cohomology of this constant presheaf. We have then proven:

Proposition 5. $E_2 = H^*(M) \otimes \left(\bigwedge \mathfrak{sl}(n, \mathbb{C})^*\right)_{\mathrm{Inv}}$.

This proposition is just a variant of the Leray's theorem for de Rham cohomology (see [1]).

The natural map $H^*(M) \to H^*(\Omega_{\mathrm{Der}}(\mathfrak{A}), \hat{d})$ defined by the inclusion $\Omega(M) \hookrightarrow \Omega_{\mathrm{Der}}(\mathfrak{A})$ is not injective. Indeed, take for example the 4-form $\mathrm{Tr}(FF)$ where F is the local expression of the curvature of an $SU(n)$-connection on E. Then we know that this form is closed, and we can write locally $\mathrm{Tr}(FF) = \mathrm{d}\,\mathrm{Tr}(A(\mathrm{d}A + \frac{2}{3}AA))$. We know that in ordinary differential geometry, those local Chern–Simon forms do not always define a global 3-form on M. But in the noncommutative geometry of $\mathfrak{A}$, we have a well-defined global 3-form $\mathrm{Tr}(\alpha(\hat{d}\alpha + \frac{2}{3}\alpha\alpha))$, where α is the noncommutative 1-form defined by the connection. This 3-form satisfies

$$\hat{d}\,\mathrm{Tr}\left(\alpha\left(\hat{d}\alpha + \tfrac{2}{3}\alpha\alpha\right)\right) = \mathrm{Tr}(FF)$$

because $\hat{d}\alpha + \alpha\alpha$ is exactly the curvature of the connection. So, the class of $\mathrm{Tr}(FF)$ is not always zero in $H^4(M)$ but is always zero in $H^4(\Omega_{\mathrm{Der}}(\mathfrak{A}), \hat{d})$. The situation is obviously the same for any other characteristic class of E. This means that the cohomology of $\Omega_{\mathrm{Der}}(\mathfrak{A})$ does not see the nontriviality of E. This situation is very similar to that encountered with the ordinary cohomology of the associated principal bundle.

7. Conclusions

In this paper, we have added a few more arguments to those given in [5], to propose the algebra $\mathfrak{A}$ equipped with its noncommutative differential calculus based on derivations as a possible replacement of the principal bundle P. Indeed, we have shown that this noncommutative geometry of $\mathfrak{A}$ is strongly and very naturally related to the ordinary geometry of P. Then $\mathfrak{A}$ can be used in place of P, if one replaces the differential calculus $\Omega(P)$ by $\Omega_{\mathrm{Der}}(\mathfrak{A})$, the connection 1-form ω on P by the associated noncommutative 1-form α introduced in [5] (which is only subjected to a "vertical" condition), the notion of associated vector bundle by the notion of (left-)module over $\mathfrak{A}$.

From a physical point of view, this noncommutative geometry is more interesting because, as was pointed out in [4,5], it contains not only ordinary Yang–Mills fields, but also other fields which look very much like Higgs fields.

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References

[1] R. Bott, L. Tu, Differential Forms in Algebraic Topology, GMT 82, Springer, Berlin, 1982.
[2] M. Dubois-Violette, Dérivations et calcul différentiel non commutatif, C.R. Acad. Sci. Paris, vol. 307, Série I, 1988, pp. 403–408.
[3] M. Dubois-Violette, R. Kerner, J. Madore, Noncommutative differential geometry of matrix algebras, J. Math. Phys. 31 (1990) 316.
[4] M. Dubois-Violette, R. Kerner, J. Madore, Noncommutative differential geometry and new models of gauge theory, J. Math. Phys. 31 (1990) 323.
[5] M. Dubois-Violette, T. Masson, $SU(n)$-connections and noncommutative differential geometry, J. Geom. Phys. 25 (1,2) (1998) 104.
[6] M. Dubois-Violette, P.W. Michor, Connections on central bimodules in noncommutative differential geometry, J. Geom. Phys. 20 (1996) 218.
[7] T. Masson, Submanifolds and quotient manifolds in noncommutative geometry, J. Math. Phys. 37 (5) (1996) 2484.